VIBRATION OF A SECOND ORDER SYSTEM WITH A RANDOMLY VARYING PARAMETER

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PMM Vol.26, No.4, 1962, pp. 762-766

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(Received February 6, 1962)

Investigation of the statistical properties of solutions of differential equations with random, time-varying parameters is a complicated task. Closed expressions for statistical characteristics (moments in particular) can be obtained only in those cases when a solution of an equation can be written as an explicite expression whose parameters are arbitrary functions of time. This indirect method of investigating the properties of solutions is used for the first order differential equations whose coefficients are random functions of time and whose integration has been carried out through the separation of variables. This kind of investigation has been applied to a certain linear equation by Tikhonov [1]. For the second order equations, however, where the coefficients are random functions of time, a general solution cannot be obtained and the indirect method of investigating the statistical properties of solutions is not possible.

By using approximate methods based on assumptions of slow variation of parameters, of small fluctuations of parameters, etc. solutions can be found for a much wider class of differential equations. This in turn permits an estimate of statistical properties of these solutions (see, for example, [2-5]).

We consider here the second order differential equation

$$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + [1 + \varepsilon_1\xi(t)]\,x = 0 \tag{1}$$

where $\xi(t)$ is a random function of the time, and ε_1 is a small parameter. The correlation function

$$R(\Delta t) = \overline{\xi(t + \Delta t)\,\xi(t)}$$

is considered to be known and the mathematical expectation of $\xi(t)$ is assumed to be equal zero

$$\overline{\xi(t)} = 0$$

When $\xi(t)$ is a periodic function of time, then we have the well known classical case of parametric resonance (see, for example, [6]). The case of $\xi(t)$ being a random function has been studied in the book by Stratonovich [2], and certain properties of solutions in this case have also been derived in [3, 4].

By the conventional substitution of new variables

$$z = e^{nt}x, \qquad \tau = t\sqrt{1+n^2}$$

Equation (1) is reduced to

$$\frac{d^2z}{d\tau^2} + z = -\varepsilon\xi(\tau)z \qquad \left(\varepsilon = \frac{\varepsilon_1}{1+n^2}\right) \tag{2}$$

It is assumed that $n \ll 1$. Since ε is a small parameter, the solution of Equation (2), will approach the harmonic solution; consequently, it is convenient to seek the solution in the form

$$z(\tau) = a(\tau)\cos\tau + b(\tau)\sin\tau \tag{3}$$

We shall initially construct an approximate solution of the well defined problem, which consists of finding $a(\tau)$ and $b(\tau)$, when $\xi(\tau)$ is an arbitrary given function of time. Applying the method of Lagrange and operating on the right-hand side as if it were a known function of time, we obtain the derivatives of the functions $a(\tau)$ and $b(\tau)$

$$\frac{da}{d\tau} = \varepsilon\xi(\tau)z(\tau)\sin\tau, \qquad \frac{db}{d\tau} = -\varepsilon\xi(\tau)z(\tau)\cos\tau \tag{4}$$

Substituting (3) on the right-hand side we obtain

$$\frac{da}{d\tau} = \varepsilon\xi(\tau)(a\sin\tau\cos\tau + b\sin^2\tau) \qquad \frac{db}{d\tau} = -\varepsilon\xi(\tau)(a\cos^2\tau + b\sin\tau\cos\tau) \tag{5}$$

Writing the expansions of $a(\tau)$ and $b(\tau)$ in powers of ε we have

$$a(\tau) = a^{(0)}(\tau) + \varepsilon a^{(1)}(\tau) + \varepsilon^2 a^{(2)}(\tau) + \ldots, \quad b(\tau) = b^{(0)}(\tau) + \varepsilon b^{(1)}(\tau) + \varepsilon^2 b^{(2)}(\tau) + \ldots \tag{6}$$

Substituting the above expansions in Equations (5) and equating the coefficients of like powers of ε, we obtain the recurrent system of equations

$$\frac{da^{(0)}}{d\tau} = 0, \qquad \frac{db^{(0)}}{d\tau} = 0$$

$$\begin{aligned}
\frac{da^{(1)}}{d\tau} &= \xi(\tau)\,(a^{(0)} \sin\tau \cos\tau + b^{(0)} \sin^2\tau) \\
\frac{db^{(1)}}{d\tau} &= -\,\xi(\tau)\,(a^{(0)} \cos^2\tau + b^{(0)} \sin\tau \cos\tau) \\
\frac{da^{(2)}}{d\tau} &= \xi(\tau)\,[a^{(1)}(\tau) \sin\tau \cos\tau + b^{(1)}(\tau) \sin^2\tau] \\
\frac{db^{(2)}}{d\tau} &= -\,\xi(\tau)\,[a^{(1)}(\tau) \cos^2\tau + b^{(1)}(\tau) \sin\tau \cos\tau] \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{7}$$

Hence

$$a^{(0)}(\tau) = a^{(0)}\,(= \text{const}), \qquad b^{(0)}(\tau) = b^{(0)}\,(= \text{const})$$

$$\begin{aligned}
a^{(1)}(\tau) &= a^{(0)} \int_{\tau_0}^{\tau} \xi(\tau_1) \sin\tau_1 \cos\tau_1\, d\tau_1 + b^{(0)} \int_{\tau_0}^{\tau} \xi(\tau_1) \sin^2\tau_1\, d\tau_1 \\
b^{(1)}(\tau) &= -\,a^{(0)} \int_{\tau_0}^{\tau} \xi(\tau_1) \cos^2\tau_1\, d\tau_1 - b^{(0)} \int_{\tau_0}^{\tau} \xi(\tau_1) \sin\tau_1 \cos\tau_1\, d\tau_1 \\
a^{(2)}(\tau) &= \int_{\tau_0}^{\tau} a^{(1)}(\tau_1)\, \xi(\tau_1) \sin\tau_1 \cos\tau_1\, d\tau_1 + \int_{\tau_0}^{\tau} b^{(1)}(\tau_1)\, \xi(\tau_1) \sin^2\tau_1\, d\tau_1 \\
b^{(2)}(\tau) &= -\int_{\tau_0}^{\tau} a^{(1)}(\tau_1)\, \xi(\tau_1) \cos^2\tau_1\, d\tau_1 - \int_{\tau_0}^{\tau} b^{(1)}(\tau_1)\, \xi(\tau_1) \sin\tau_1 \cos\tau_1\, d\tau_1 \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{8}$$

We shall next consider the values of $z(\tau)$ and consequently the values of $a(\tau)$ and $b(\tau)$ in one period intervals, that is at the following instants of time: $\tau = 0,\ 2\pi,\ \ldots,\ 2\pi(i-1),\ 2\pi i,\ \ldots$. Let us see what happens when the instant of time changes from $\tau = 2\pi(i-1)$ to $\tau = 2\pi i$. Let

$$a\,|_{\tau=2\pi(i-1)} = a_{i-1} = a^{(0)}$$

$$b\,|_{\tau=2\pi(i-1)} = b_{i-1} = b^{(0)}, \qquad a^{(k)}\,|_{\pi=2\pi(i-1)} = b^{(k)}\,|_{\tau=2\pi(i-1)} = 0 \qquad (k = 1, 2, \ldots)$$

Then

$$a_i = a\,|_{\tau=2\pi i} = a_i^{(0)} + \varepsilon a_i^{(1)} + \varepsilon^2 a_i^{(2)} + \ldots, \quad b_i = b\,|_{\tau=2\pi i} = b_i^{(0)} + \varepsilon b_i^{(1)} + \varepsilon^2 b_i^{(2)} + \ldots$$

Here

$$a_i^{(0)} = a_{i-1}, \qquad b_i^{(0)} = b_{i-1} \tag{9}$$

$$a_i^{(1)} = a_i^{(0)} \int_0^{2\pi} \xi\,[2\pi(i-1) + \tau_1] \sin \tau_1 \cos \tau_1\, d\tau_1 + b_i^{(0)} \int_0^{2\pi} \xi\,[2\pi(i-1) + \tau_1] \sin^2 \tau_1\, d\tau_1$$

$$b_i^{(1)} = -a_i^{(0)} \int_0^{2\pi} \xi\,[2\pi(i-1) + \tau_1] \cos^2 \tau_1\, d\tau_1 - b_i^{(0)} \int_0^{2\pi} \xi\,[2\pi(i-1) + \tau_1] \sin \tau_1 \cos \tau_1\, d\tau_1$$

. . . .

We shall now examine statistical properties of the solutions when $\xi(\tau)$ is a random function of time. The initial conditions at $\tau = 2\pi(i - 1)$ are also assumed to be random and the mathematical expectation of a_{i-1} and b_{i-1} will be zero

$$\bar{a}_{i-1} = \bar{b}_{i-1} = 0 \tag{10}$$

It is obvious that in this case we shall get

$$\bar{a}_i = \bar{b}_i = 0 \tag{11}$$

which would remain valid for any i. Averaging over the set we obtain for the second correlation moments of a_i and b_i

$$\begin{aligned} \overline{a_i^2} &= \overline{a_i^{(0)2}} + 2\varepsilon \overline{a_i^{(0)} a_i^{(1)}} + \varepsilon^2 (\overline{a_i^{(1)2}} + 2\overline{a_i^{(0)} a_i^{(2)}}) + \ldots \\ \overline{b_i^2} &= \overline{b_i^{(0)2}} + 2\varepsilon \overline{b_i^{(0)} b_i^{(1)}} + \varepsilon^2 (\overline{b_i^{(1)2}} + 2\overline{b_i^{(0)} b_i^{(2)}}) + \ldots \\ \overline{a_i b_i} &= \overline{a_i^{(0)} b_i^{(0)}} + \varepsilon (\overline{a_i^{(0)} b_i^{(1)}} + \overline{b_i^{(0)} a_i^{(1)}}) + \varepsilon^2 (\overline{a_i^{(0)} b_i^{(2)}} + \overline{a_i^{(1)} b_i^{(1)}} + \overline{b_i^{(0)} a_i^{(2)}}) \end{aligned} \tag{12}$$

At any given stage the initial conditions $a_i^{(0)}$ and $b_i^{(0)}$ and the coefficients $a_i^{(k)}$ and $b_i^{(k)}$ $(k = 1, 2, \ldots)$ will be not, in general, uncorrelated. We shall assume further that the correlation interval of $\xi(\tau)$ is considerably smaller than the period. Taking into account (11) we obtain

$$\overline{a_i^{(0)} a_i^{(1)}} = \overline{a_i^{(0)}}\; \overline{a_i^{(1)}} = 0 \tag{13}$$

Consequently, the systems of Equations (12) can be written in the form

$$\overline{a_i^2} = \overline{a_i^{(0)2}} + \varepsilon^2 \overline{a_i^{(1)2}} + 2\overline{a_i^{(0)} a_i^{(2)}}) + \ldots, \qquad \overline{b_i^2} = \overline{b_i^{(0)2}} + \varepsilon^2 (\overline{b_i^{(1)2}} + 2\overline{b_i^{(0)} b_i^{(2)}}) + \ldots$$

$$\overline{a_i b_i} = \overline{a_i^{(0)} b_i^{(0)}} + \varepsilon^2 (\overline{a_i^{(0)} b_i^{(2)}} + \overline{a_i^{(b)} b_i^{(1)}} + \overline{b_i^{(0)} b_i^{(2)}}) + \ldots \tag{14}$$

Retaining only the second order terms with respect to ε, the above equations, in the expanded form, will be

$$\begin{aligned}
\overline{a_i^{\,2}} &= (1+\varepsilon^2 J_1)\,\overline{a_{i-1}^2} + \varepsilon^2 J_2 \overline{b_{i-1}^2} + \varepsilon^2 J_3 \overline{a_{i-1} b_{i-1}} \\
\overline{b_i^{\,2}} &= \varepsilon^2 J_4 \overline{a_{i-1}^2} + (1+\varepsilon^2 J_1)\,\overline{b_{i-1}^2} + \varepsilon^2 J_5 \overline{a_{i-1} b_{i-1}} \\
\overline{a_i b_i} &= -\frac{1}{2}\varepsilon^2 J_5 \overline{a_{i-1}^2} + -\frac{1}{2}\varepsilon^2 J_3 \overline{b_{i-1}^2} + (1+\varepsilon^2 J_6)\,\overline{a_{i-1} b_{i-1}}
\end{aligned} \tag{15}$$

Here

$$J_1 = 3J'_1 - 2J'_6, \qquad J'_1 = \int_0^{2\pi} \sin\tau_1 \cos\tau_1 \, d\tau_1 \int_0^{2\pi} \sin\tau_2 \cos\tau_2 R(\tau_1 - \tau_2)\, d\tau_2$$

$$J_6 = J'_1 - 3J'_6, \qquad J_2 = \int_0^{2\pi} \sin^2\tau_1 \, d\tau_1 \int_0^{2\pi} \sin^2\tau_2 R(\tau_1 - \tau_2)\, d\tau_2$$

$$J_3 = 2\int_0^{2\pi} \sin\tau_1 \cos\tau_1 \, d\tau_1 \int_0^{2\pi} \sin^2\tau_2 R(\tau_1 - \tau_2)\, d\tau_2$$

$$J_4 = \int_0^{2\pi} \cos^2\tau_1 \, d\tau_1 \int_0^{2\pi} \cos^2\tau_2 R(\tau_1 - \tau_2)\, d\tau_2, \qquad J_6' = \int_0^{2\pi} \sin^2\tau_1 \, d\tau_1 \int_0^{2\pi} \cos^2\tau_2 R(\tau_1 - \tau_2)\, d\tau_2$$

$$J_5 = 2\int_0^{2\pi} \sin\tau_1 \cos\tau_1 \, d\tau_1 \int_0^{2\pi} \cos^2\tau_2 R(\tau_1 - \tau_2)\, d\tau_2 \tag{16}$$

The system (15) is a system of finite difference equations with respect to the correlation moments $a_i^{\,2}$, $b_i^{\,2}$ and $a_i b_i$. Since $\xi(\tau)$ is a stationary random function, the expressions $J_1, \ldots, J_6$ are independent of i. The solution in this case [7] is of the form

$$\overline{a_i^{\,2}} = \eta \overline{a_{i-1}^2}, \qquad \overline{b_i^{\,2}} = \eta \overline{b_{i-1}^2}, \qquad \overline{a_i b_i} = \eta \overline{a_{i-1} b_{i-1}} \tag{17}$$

The characteristic values η_1, η_2 and η_3 are the roots of the equation

$$\begin{vmatrix}
-\eta + 1 + \varepsilon^2 J_1 & \varepsilon^2 J_2 & \varepsilon^2 J_3 \\
\varepsilon^2 J_4 & -\eta + 1 + \varepsilon^2 J_1 & \varepsilon^2 J_5 \\
-\frac{1}{2}\varepsilon^2 J_5 & -\frac{1}{2}\varepsilon^2 J_3 & -\eta + 1 + \varepsilon^2 J_6
\end{vmatrix} = 0 \tag{18}$$

If the second order terms (with the multiplier ε^2) are neglected, then $\eta_1 = \eta_2 = \eta_3 = 1$, which means that $\xi(\tau)$ does not influence the stability of the system controlled by Equations (2). This latter result can also be derived from the formulas of Stratonovich [2]. When $\xi(\tau)$ is not a narrow band signal with slowly or little changing phase, then, within the first order terms, damping remains constant.

In the case under consideration (within the second order terms) we obtain $\eta_1 \neq \eta_2 \neq \eta_3 \neq 1$. The statistical properties of the solutions of

(1) are determined from the transformed characteristic values

$$\xi = \eta \exp\left(-\frac{2\pi n}{\sqrt{1+n^2}}\right) \approx \eta\,(1-2\pi n) \tag{19}$$

In order that the solution is stable it is necessary that all $|\xi_i| < 1$ $(i = 1, 2, 3)$.

For the case when the correlation function of the variable $\xi(\tau)$ is

$$R(\Delta\tau) = e^{-\beta|\Delta\tau|} \tag{20}$$

we can carry out calculations to the end.

Calculating the integrals (16) we obtain

$$\begin{aligned}
J_1' &= \frac{1}{2(\beta^2+4)}\left[\pi\beta + \frac{4}{\beta^2+4}(1-e^{-2\pi\beta})\right]\\
J_2 &= \frac{1}{2\beta(\beta^2+4)}\left[\pi(3\beta^2+8) - \frac{16}{\beta(\beta^2+4)}(1-e^{-2\pi\beta})\right]\\
J_3 &= -\frac{2}{\beta^2+4}\left[\pi + \frac{4}{\beta^2}(1-e^{-2\pi\beta})\right]\\
J_4 &= \frac{1}{2\beta(\beta^2+4)}\left[\pi(3\beta^2+8) - \frac{4(\beta^2+2)^2}{\beta(\beta^2+4)}(1-e^{-2\pi\beta})\right]\\
J_5 &= \frac{2}{\beta^2+4}\left[\pi - \frac{2(\beta^2+2)}{\beta^2+4}(1-e^{-2\pi\beta})\right]\\
J_6' &= \frac{1}{2\beta}\left[\pi - \frac{8(\beta^2+2)}{\beta(\beta^2+4)^2}(1-e^{-2\pi\beta})\right]
\end{aligned} \tag{21}$$

As has been mentioned earlier, the interval of correlation of $\xi(\tau)$ is assumed to be small, that is $\beta \gg 1$. For this reason we shall retain in Expressions (21) only terms of the first order with respect to β^{-1}. We shall get then

$$J_1 \approx \frac{\pi}{2\beta}, \qquad J_2 \approx J_4 \approx \frac{3\pi}{2\beta}, \qquad J_6 \approx -\frac{\pi}{\beta}, \qquad J_3 = J_5 \approx 0 \tag{22}$$

It is easily seen that the last equation of the system (15) can be solved independently of the remaining two. The corresponding characteristic value equals

$$\eta_1 \approx 1 - \varepsilon^2\frac{\pi}{\beta} \tag{23}$$

The remaining two equations of the system (15) are coupled. The corresponding characteristic values are

$$\eta_{2,3} \approx 1 + \varepsilon^2\frac{\pi}{2\beta}(1 \pm \sqrt{3}) \tag{24}$$

The transformed characteristic value largest in absolute value

$$\xi_3 \approx 1 - 2\pi n + \varepsilon^2 \frac{2\pi}{2\beta}(1+\sqrt{3}) \tag{25}$$

determines the statistical stability of the solution of (1).

When $\xi_3 < 1$, that is when $n\beta\varepsilon^{-2} > 0.68$, the mean square of the amplitude is decreasing and the solutions of (1) are statistically stable.

When $\xi_3 > 1$, that is when $n\beta\varepsilon^{-2} < 0.68$, the mean square of the amplitude grows without bounds and the solutions are unstable.

Let us mention that for the transformed characteristic value corresponding to the mutual correlation moment $a_i b_i$, whether the condition of stability is satisfied or not, we have always

$$\xi_1 = 1 - 2\pi n - \varepsilon^2 \frac{\pi}{\beta} < 1$$

This means that the correlation between the orthogonal components is decreasing with time, that is after a sufficiently great lapse of time the phase will be equally probable, becoming independent of its initial distribution.

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Translated by T.L.